Math 564: Advance Analysis 1
Lecture 20

Lebesgue Differentiation Theorem. For each locally integrable $f: \mathbb{R}^d \to \mathbb{R}$ [i.e. $f \cdot \mathbb{1}_B \in L^1(\mathbb{R}^d, \lambda)$ for every bounded box B), we have:

$$f(x) = \lim_{r \to 0} \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda$$

for a.e. $x \in \mathbb{R}^d$, where $B_r(x)$ is the ball of radius r about x in d_∞-metric.

In English, this theorem says that almost surely, the value of the function f at x is equal the limit of averages of values of f around x.

Motivation. Let L^1_{loc} denote the set of all locally integrable functions f, i.e. f s.t. $f \cdot \mathbb{1}_B \in L^1$ for all balls $B \subseteq \mathbb{R}^d$.

Define the averaging operator $A_r : L^1_{loc} \to L^1_{loc}$ by $A_r f(x) := \int_{B_r(x)} f \, d\lambda / \lambda(B_r(x))$.

Lemma 1. If $g: \mathbb{R}^d \to \mathbb{R}$ is continuous and locally integrable, then

$$g(x) = \lim_{r \to 0} A_r g(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Proof. Fix $x \in \mathbb{R}^d$.

Then
$$\begin{aligned}
|A_r g(x) - g(x)| &= \left| \frac{1}{\lambda(B_r)} \int_{B_r(x)} g(y) \, d\lambda(y) - g(x) \right| \\
&= \left| \frac{1}{\lambda(B_r)} \int_{B_r(x)} g(y) \, d\lambda(y) - \frac{1}{\lambda(B_r)} \int_{B_r(x)} g(x) \, d\lambda(y) \right| \\
&\leq \frac{1}{\lambda(B_r)} \int_{B_r(x)} |g(y) - g(x)| \, d\lambda(y) \\
[\text{for small enough } r>0] \quad &\leq \frac{1}{\lambda(B_r)} \int_{B_r(x)} \varepsilon \, d\lambda(y) = \varepsilon.
\end{aligned}$$

$\square$

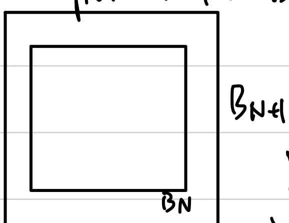
Local-global bridge. Let f be in L^1 and $r>0$.

(a) $\int f \, d\lambda = \int A_r f \, d\lambda$, average of the local averages is the average.

(b) $\|A_r f\|_1 \leq \|f\|_1$, in fact $\|A_r |f|\| = \|f\|_1$.

Proof. (a) is by Fubini and (b) follows from (a), left as HW. $\square$

Proof of Lebesgue Differentiation. We may assume WLOG that f is in L^1 by replacing f with $f \mathbb{1}_B$ for all balls B. Indeed, if the theorem holds for $f \cdot \mathbb{1}_{B_{N+1}}$ where $B_k :=$ ball at the origin of radius k, then $A_r(f \cdot \mathbb{1}_{B_{N+1}}) = A_r f$ for all points in B_N, $\forall r \leq 1$.



For each $\alpha \geq 0$, let $D_\alpha := \{x \in X : |\overbrace{\limsup_{r \to 0} A_r f(x)}^{A_* f} - f(x)| > \alpha\}$. We want to show that D_0 is null, which it is enough to show that D_α is null for all $\alpha > 0$ because $D_0 = \bigcup_{n \in \mathbb{N}^+} D_{\frac{1}{n}}$.

Fix $\alpha > 0$ and $\varepsilon > 0$ to show that $\lambda(D_\alpha) \leq \varepsilon$.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function $\in L^1$ with $\|f-g\|_1 < \delta$, where δ will be chosen later (depending on ε and α, but not on g). For all $x \in \mathbb{R}^d$,

$$|A_* f(x) - f(x)| = |A_* f(x) - A_* g(x) + A_* g(x) - g + g - f| \leq |A_* f(x) - A_* g(x)| + \underbrace{|A_* g(x) - g(x)|}_{=0} + |g(x) - f(x)| = |A_*(f-g)(x)| + |f(x) - g(x)|.$$

$$\text{Thus, } D_\alpha = \Delta_\alpha(A_* f - f) \subseteq \overset{(b)}{\Delta_{\alpha/2}(A_*(f-g))} \cup \overset{(a)}{\Delta_{\alpha/2}(f-g)}.$$

(a) By Chebyshev: $\frac{\alpha}{2} \cdot \lambda(\Delta_{\alpha/2}(f-g)) \leq \|f-g\|_1 < \delta$, so take $\delta \leq \frac{\varepsilon \cdot \alpha}{4}$.

$$\text{so } \lambda(\Delta_{\alpha/2}(f-g)) < \frac{2}{\alpha} \cdot \delta \leq \varepsilon/2.$$

(b) We wish the following was also true: $\frac{\alpha}{2} \cdot \lambda(\Delta_{\alpha/2} A_*(f-g)) \leq \|f-g\|_1$.

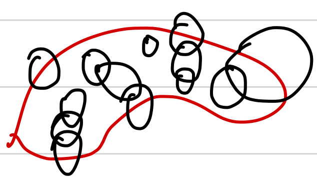
In fact it would be enough to have this with $\|f-g\|_1$ replaced with $C\cdot\|f-g\|_1$, where C is an honest constant.

Hardy–Littlewood Maximal Theorem. Let $h \in L^1$, $\alpha > 0$. Then $\alpha \cdot \lambda(\Delta_\alpha(A_* h)) \le 3^d \|h\|_1$.
In fact, $\alpha \cdot \lambda(\Delta_\alpha(\bar{A} h)) \le 3^d \|h\|_1$, where $\bar{A} h(x) := \sup_{r \le 1} A_r |h|$, called the **Hardy-Littlewood maximal function.**

Given this, the proof is over by choosing $\delta := \frac{\varepsilon \cdot \alpha}{4 3^d}$, because then
$\frac{\alpha}{2} \cdot \lambda(\Delta_{\alpha/2} A_*(f-g)) \le 3^d \|f-g\|_1 < 3^d \cdot \delta$, so
$\lambda(\Delta_{\alpha/2} A_*(f-g)) \le 3^d \cdot \frac{2}{\alpha} \cdot \delta = \varepsilon/2$. $\square$

Proof of Hardy-Littlewood Maximal Theorem. For every $x \in \Delta_\alpha(\bar{A} h) = \Delta_\alpha\left(\sup_{r \le 1} A_r h\right) =: A$, there is a radius $r_x \le 1$ s.t. $\int_{B_{r_x}(x)} |h| \, d\lambda > \alpha \cdot \lambda(B_{r_x})$. Thus, there A is covered by the collection of balls $\mathcal{C} := \{B_{r_x}(x) : x \in A\}$.

If we had $\int_A |h| \, d\lambda > \alpha \, \lambda(A)$, we would be done.

A 

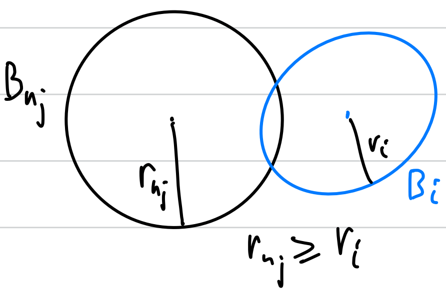
If these balls in $\mathcal{C}$ were disjoint, and ctbly many we would have this by the linearity of integrals and ctbl additivity.

Vitali Covering Lemma. Let $A \subseteq \mathbb{R}^d$ a λ-measurable set of positive measure and let $\mathcal{C}$ be a cover of A with balls. For each $a < \lambda(A)$, $\exists$ finite $\mathcal{C}_a \subseteq \mathcal{C}$ of pairwise disjoint balls such that $\sum_{B \in \mathcal{C}_a} \lambda(B) \ge a/3^d$.

Proof. Let N be large enough s.t. $\lambda(A \cap B_N) > a$. Thus we may assume A is bdd. By tightness (for finite measures), $\exists$ compact $K \subseteq A$ s.t. $\lambda(K) > a$. Thus, WLOG, we assume A is compact. Then $\mathcal{C}$ has a finite subcover of A, so WLOG, we assume $\mathcal{C}$ is finite.
Order $\mathcal{C}$ by radius of balls in decreasing order: $B_1, B_2, B_3, \dots B_k$.

We can a greedy algorithm and get a subsequence $B_{n_1} B_{n_2} B_{n_3} \cdots B_{n_\ell}$ where $n_1 := 1$, $n_2 :=$ the smallest s.t. B_{n_2} is disjoint from B_{n_1}, $n_3 :=$ the smallest s.t. B_{n_3} is disjoint from $B_{n_1} \cup B_{n_2}$, etc.

For a ball B centered at x, let $B^* :=$ the ball centered at x of radius $3 \cdot \text{radius}(B)$. Then note that $\bigcup_{i \le \ell} B^*_{n_i} \supseteq \bigcup \mathcal{C} = \bigcup_{B \in \mathcal{C}} B$ hence for each B_i where $i \neq n_j \ \forall j$, $\exists\, n_j < i$ s.t. $B_i \cap B_{n_j} \neq \emptyset$: then $B_i \subseteq B^*_{n_j}$.



Then $\lambda\left(\bigsqcup_{i=1}^{\ell} B_{n_i}\right) \geq \frac{1}{3^d} \lambda\left(\bigcup_{i=1}^{\ell} B^*_{n_i}\right) \geq \frac{1}{3^d} \lambda\left(\bigcup_{B \in \mathcal{C}} B\right) > \frac{1}{3^d} \cdot a.$ $\square$

Back to maximal theorem. Letting $a < \lambda(A)$ be arbitrary, apply Vitali covering lemma to get a disjoint collection $\mathcal{C}_a \subseteq \mathcal{C}$ s.t. $\sum_{B \in \mathcal{C}_a} \lambda(B) > \frac{1}{3^d} \cdot a$.

Then $\|f\|_1 \geq \int_A |f| \, d\lambda \geq \int_{\bigcup \mathcal{C}_a} |f| \, d\lambda = \sum_{B \in \mathcal{C}_a} \int_B |f| \, d\lambda > \alpha \cdot \sum_{B \in \mathcal{C}_a} \lambda(B) > \frac{1}{3^d} \cdot \alpha \cdot a$,

so letting $a \to \lambda(A)$, we get $\|f\|_1 \geq \frac{1}{3^d} \cdot \alpha \cdot \lambda(A) = \frac{1}{3^d} \cdot \alpha \cdot \lambda(\Delta_\alpha \bar{A} h)$. $\square$